(4) $u_k(t) = \log \frac{p(z_k=1 \mid \vec{z}_{\backslash k})}{p(z_k=0 \mid \vec{z}_{\backslash k})} \triangleq u_k$

- odds $\triangleq$ $\frac{\text{发生}}{\text{不发生}}$ 非 probability
- log-odds 即对 odds 取对数

(5) 玻尔兹曼分布 $p(\vec{z}) = \left[\sum_{v\in S} e^{-E(v)}\right]^{-1} e^{-E(\vec{z})}$ $Z = \sum_{v\in S} e^{-E(v)}$

$$p(\vec{z}) = \frac{1}{Z} \exp\left(\sum_{i,j} \frac{1}{2} W_{ij} z_i z_j + \sum_i b_i z_i\right)$$

(6) 将 (5) 代入 (4)

$$u_k(\vec{z}) = \log \frac{p(z_k=1, \vec{z}_{\backslash k}) / \cancel{p(\vec{z}_{\backslash k})}}{p(z_k=0, \vec{z}_{\backslash k}) / \cancel{p(\vec{z}_{\backslash k})}} = \log \frac{\exp\left(\sum_{i,j\neq k} \frac{1}{2} W_{ij} z_i z_j + \sum_i W_{ik} z_i + \sum_{i\neq k} b_i z_i + b_k\right)}{\exp\left(\sum_{i,j\neq k} \frac{1}{2} W_{ij} z_i z_j + \sum_{i\neq k} b_i z_i\right)}$$

$$= \sum_{i,j\neq k} \frac{1}{2} W_{ij} z_i z_j + \sum_i W_{ik} z_i + \sum_{i\neq k} b_i z_i + b_k - \left(\sum_{i,j\neq k} \frac{1}{2} W_{ij} z_i z_j + \sum_{i\neq k} b_i z_i\right)$$

$$= \sum_i W_{ik} z_i + b_k$$

$\zeta \in \{0, 1, \dots, \tau\}^K$

(7) $p(\zeta_k \mid z_k) := \begin{cases} \tau^{-1} & z_k = 1 \wedge \zeta_k > 0 \\ 1 & z_k = 0 \wedge \zeta_k = 0 \\ 0 & \text{other} \end{cases}$

$$p(\vec{\zeta} \mid \vec{z}) = \prod_k p(\zeta_k \mid z_k) \qquad p(\vec{\zeta}, \vec{z}) = p(\vec{\zeta} \mid \vec{z})\, p(\vec{z})$$

lemma 1 $p(\zeta_k \mid \vec{\zeta}_{\backslash k}, \vec{z}_{\backslash k}) = p(\zeta_k \mid \vec{z}_{\backslash k}) = \begin{cases} \sigma(u_k)/\tau & \zeta_k > 0 \\ 1 - \sigma(u_k) & \text{其他} \end{cases}$

$$p(z_k \mid \vec{\zeta}, \vec{z}_{\backslash k}) = p(z_k \mid \zeta_k) = \begin{cases} 1 & \zeta_k > 0 \wedge z_k = 1 \\ 1 & \zeta_k = 0 \wedge z_k = 0 \\ 0 & \text{other} \end{cases}$$

其中 $\sigma(x) = (1+e^{-x})^{-1}$, logit 反函数

- $p(z_k=1|\vec{z}_{\setminus k}) = 1 - p(z_k=0|\vec{z}_{\setminus k}) \hat{=} p$ $\qquad p(z_k=1|\vec{z}_{\setminus k}) = \frac{p(z_k=1,\vec{z}_{\setminus k})}{p(\vec{z}_{\setminus k})} \overset{(6)}{=} \frac{p(z_k=1,\vec{z}_{\setminus k})}{p(z_k=0,\vec{z}_{\setminus k})+p(z_k=1,\vec{z}_{\setminus k})}$
- $u_k = \log\frac{p}{1-p} \Rightarrow p = \frac{e^{u_k}}{1+e^{u_k}} = \sigma(u_k)$

1.1 $p(\zeta_k|\vec{\zeta}_{\setminus k},\vec{z}_{\setminus k}) = \sum_{z_k} p(\zeta_k, z_k|\vec{\zeta}_{\setminus k},\vec{z}_{\setminus k}) = \sum_{z_k}\frac{p(\vec{\zeta},\vec{z})}{p(\vec{\zeta}_{\setminus k},\vec{z}_{\setminus k})} = \sum_{z_k}\frac{p(\vec{\zeta}|\vec{z})p(\vec{z})}{p(\vec{\zeta}_{\setminus k}|\vec{z}_{\setminus k})p(\vec{z}_{\setminus k})}$

$$= \sum_{z_k}\frac{\left[\prod_{l\neq k} p(\zeta_l|z_l)\right] p(\zeta_k|z_k)\, p(\vec{z})}{\prod_{l\neq k} p(\zeta_l|z_l)\, p(\vec{z}_{\setminus k})} = \sum_{z_k} p(\zeta_k|z_k)\, p(z_k|\vec{z}_{\setminus k})$$

$$= \underbrace{p(\zeta_k|z_k=1)}_{\chi_{\{1,\dots,\tau\}}(\zeta_k)\tau^{-1}}\underbrace{p(z_k=1|\vec{z}_{\setminus k})}_{\sigma(u_k)} + \underbrace{p(\zeta_k|z_k=0)}_{\delta(\zeta_k,0)}\underbrace{p(z_k=0|\vec{z}_{\setminus k})}_{1-\sigma(u_k)}$$

ζ_k 与 $\vec{\zeta}_{\setminus k}$ 关于 $\vec{z}_{\setminus k}$ 条件独立

1.2 $p(z_k|\vec{\zeta},\vec{z}_{\setminus k}) = \frac{p(z_k,\zeta_k|\vec{\zeta}_{\setminus k},\vec{z}_{\setminus k})}{p(\zeta_k|\vec{\zeta}_{\setminus k},\vec{z}_{\setminus k})} = \frac{p(\zeta_k|\vec{\zeta}_{\setminus k},\vec{z})\,p(z_k|\vec{\zeta}_{\setminus k},\vec{z}_{\setminus k})}{p(\zeta_k|\vec{\zeta}_{\setminus k},\vec{z}_{\setminus k})}$

注: $p(A|B,C) = \frac{p(A,B|C)}{p(B|C)}$ $\qquad p(A,B|C) = p(A|B,C)\,p(B|C)$

$p(z_k|\vec{z}_{\setminus k},\vec{\zeta}_{\setminus k}) = p(z_k|\vec{z}_{\setminus k})$ (条件独立)

$$p(\zeta_k|\vec{\zeta}_{\setminus k},\vec{z}) = p(\zeta_k|\vec{z}) = \begin{cases}\tau^{-1} & z_k=1,\ \zeta_k>0\\ 1 & z_k=0,\ \zeta_k=0\end{cases}$$

$$= z_k\,\chi_{\{1,\dots,\tau\}}(\zeta_k)\tau^{-1} + (1-z_k)\,\delta(0,\zeta_k)$$

$$\therefore p(z_k|\vec{\zeta},\vec{z}_{\setminus k}) = \frac{z_k\,\chi_{\{1,\dots,\tau\}}(\zeta_k)\tau^{-1} + (1-z_k)\,\delta(0,\zeta_k)}{\sigma(u_k)\chi_{\{1,\dots,\tau\}}(\zeta_k)\tau^{-1} + (1-\sigma(u_k))\,\delta(0,\zeta_k)}\, p(z_k|\vec{z}_{\setminus k},\vec{\zeta}_{\setminus k})$$

$$= \begin{cases} z_k & \zeta_k>0\\ 1-z_k & \zeta_k=0\end{cases} \qquad (p(z_k|\vec{z}_{\setminus k}) = \sigma(u_k))$$

算子：

$T := T^1 \cdot \cdots \cdot T^k$

T^k：只对第k个神经元作用

$$T^k(\vec{\zeta}, \vec{z} \mid \vec{\zeta}', \vec{z}') = T^k(\zeta_k, z_k \mid \vec{\zeta}', \vec{z}')\, \delta(\vec{\zeta}_{\setminus k}, \vec{\zeta}'_{\setminus k})\, \delta(\vec{z}_{\setminus k}, \vec{z}'_{\setminus k})$$

$$T^k(\zeta_k, z_k \mid \vec{\zeta}', \vec{z}') := \underbrace{\delta(z_k, \zeta_k^{>0})}_{\substack{z_k=1,\ \zeta_k>0 \\ z_k=0,\ \zeta_k=0}} \cdot T^k(\zeta_k \mid \zeta'_k, \vec{z}'_{\setminus k})$$

$\left.\begin{matrix} z_k=1,\ \zeta_k>0 \\ z_k=0,\ \zeta_k=0 \end{matrix}\right\}$ 才可能转移

$$\zeta_k^{>0} = \begin{cases} 1 & \zeta_k > 0 \\ 0 & \zeta_k = 0 \end{cases}$$

$$T^k(\zeta_k \mid \zeta'_k, \vec{z}'_{\setminus k}) = \begin{cases} \sigma(u'_k - \log\tau) & \zeta_k = \tau,\ \zeta'_k = 0,1 \\ 1 - \sigma(u'_k - \log\tau) & \zeta_k = 0,\ \zeta'_k = 0,1 \\ 1 & \zeta_k = \zeta'_k - 1,\ \zeta'_k > 1 \\ 0 & \end{cases}$$

不应期以外 fire 与不 fire ~~概率~~ 和为 1

$$u'_k = \operatorname{logit}\left(p(z_k=1 \mid \vec{z}'_{\setminus k})\right) = \log \frac{p(z_k=1 \mid \vec{z}'_{\setminus k})}{p(z_k=0 \mid \vec{z}'_{\setminus k})}$$

及 $\sigma(u'_k) = p(z_k=1 \mid \vec{z}'_{\setminus k})$

Q：T^k？

$$\sigma(u'_k - \log\tau) = \frac{e^{u'_k - \log\tau}}{1 + e^{u'_k - \log\tau}} = \frac{\frac{1}{\tau} e^{u'_k}}{1 + \frac{1}{\tau} e^{u'_k}}$$

- 目的： 转移概率 $p(\zeta_k = \tau \mid \zeta'_k = 0, 1, \vec{z}'_{\setminus k})$ 即 fire 的概率

注：$p(A \mid B \cup C) \overset{B \cap C = \phi}{=\!=} \dfrac{p(A, B \cup C)}{p(B \cup C)} = \dfrac{p(A \cap B,\ A \cap C)}{p(B \cup C)}$ $\quad (A \cap B) \cap (A \cap C) = \phi$

$$= \frac{p(A \mid B)\, p(B)}{p(B) + p(C)} + \frac{p(A \mid C)\, p(C)}{p(B) + p(C)}$$

已知 $T^k(\vec{\zeta}, \vec{z} \mid \vec{\zeta}', \vec{z}') = T^k(\zeta_k, z_k \mid \vec{\zeta}', \vec{z}')\, \delta(\vec{\zeta}'_{\setminus k}, \vec{\zeta}_{\setminus k})\, \delta(\vec{z}'_{\setminus k}, \vec{z}_{\setminus k})$

故只考虑 $\vec{\zeta}'_{\setminus k} = \vec{\zeta}_{\setminus k}$ 及 $\vec{z}'_{\setminus k} = \vec{z}_{\setminus k}$ 情况

$$T^k\left(\zeta_k=\tau \,\middle|\, \zeta_k'=0\text{或}1,\ \vec{z}_{\backslash k}\right)=\frac{p(\zeta_k=\tau \mid \zeta_k', \vec{z}_{\backslash k})}{p(\zeta_k=0 \mid \zeta_k', \vec{z}_{\backslash k})+p(\zeta_k=\tau \mid \zeta_k', \vec{z}_{\backslash k})}$$

$$=\frac{\sum_{z_k} p(\zeta_k=\tau, z_k \mid \zeta_k'=0\text{或}1,\ \vec{z}_{\backslash k})}{\sum_{z_k}\left[p(\zeta_k=\tau, z_k \mid \zeta_k'=0\text{或}1,\ \vec{z}_{\backslash k})+p(\zeta_k=0, z_k \mid \zeta_k'=0\text{或}1,\ \vec{z}_{\backslash k})\right)}$$

$$=\frac{p(\zeta_k=\tau \mid z_k=1, \zeta_k'=0\text{或}1, \vec{z}_{\backslash k})\,p(z_k=1 \mid \zeta_k'=0\text{或}1, \vec{z}_{\backslash k})+\cancelto{0}{p(\zeta_k=\tau \mid z_k=0, \cdots)}\,p(z_k=0 \mid \cdots)}{p(\zeta_k=\tau \mid z_k=1, \cdots)\,p(z_k=1 \mid \cdots)+\cancelto{0}{p(\zeta_k=\tau \mid z_k=0, \cdots)}\,p(z_k=0 \mid \cdots)+\cancelto{0}{p(\zeta_k=0 \mid z_k=1, \cdots)}\,p(z_k=1 \mid \cdots)+p(\zeta_k=0 \mid z_k=0 \cdots)\,p(z_k=0 \mid \cdots)}$$

$$=\frac{\frac{1}{\tau}\,p(z_k=1 \mid \vec{z}_{\backslash k})}{\frac{1}{\tau}\,p(z_k=1 \mid \vec{z}_{\backslash k})+1\cdot p(z_k=0 \mid \vec{z}_{\backslash k})}=\frac{\frac{1}{\tau}e^{u_k'}}{\frac{1}{\tau}e^{u_k'}+1}=\sigma(u_k'-\log\tau)$$

注：由 $u_k(t)$ 定义 $\quad u_k=\log\dfrac{p(z_k=1 \mid \vec{z}_{\backslash k})}{p(z_k=0 \mid \vec{z}_{\backslash k})} \Rightarrow e^{u_k}=\dfrac{p(z_k=1 \mid \vec{z}_{\backslash k})}{p(z_k=0 \mid \vec{z}_{\backslash k})}$

猜想：为什么 $T^k(\zeta_k \mid \zeta_k', \vec{z}_{\backslash k})$ 与 $p(\vec{z})$ 无关？ $\longrightarrow$ 条件概率.

$$p(\vec{z})=\sum_{\vec{\zeta}} p(\vec{\zeta}, \vec{z})$$

lemma 2 $\quad p(\zeta_k \mid \vec{z}_{\backslash k})$ 关于 $T^k(\zeta_k \mid \zeta_k', \vec{z}_{\backslash k})$ 平稳 $\quad k=1,2,\ldots,K$

lemma 3 $\quad p(\vec{\zeta}, \vec{z})$ 关于 $T^k(\vec{\zeta}, \vec{z} \mid \vec{\zeta}', \vec{z}')$ 平稳 $\quad k=1,2,\ldots,K$

Thm 不可约.非周期 Markov chain，平稳分布存在且唯一

$\sigma(x)\,(1-\sigma(x))^{-1}=e^x \qquad 1-\sigma(x)=\sigma(-x)$

总结：

$\forall$ 分布 $p(\vec{z})$ 给定，如何 neural sampling？

$$u_k(t) = \log \frac{p(z=1 \mid \vec{z}_{\setminus k})}{p(z=0 \mid \vec{z}_{\setminus k})}$$ + 给定 T $\Rightarrow$ 构建 T.

此时 $p(\vec{\xi}, \vec{z})$ 为 T 平稳分布. 可以用 MCMC

$$u_k = \log \frac{p(z=1)}{p(z=0)}$$

$$u_k = b_i + \sum_i W_{ik} \cancel{z} \, z_i$$

$$e^{u_k} = \frac{p(z_k=1)}{p(z_k=0)}$$

$$\Rightarrow \quad p(z)$$